A Theory of the Casimir Effect for Compact Regions

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Abstract

We develop Dietz's idea of identifying the Casimir energy as the regularization-independent Ramanujan sum of an asymptotic series. We also provide for the first time a solution (in this framework) of the external problem for the cube. The complete analytic and rigorous derivation of the Casimir effect for the cube permits an identification of the features which are responsible for repulsive sign of the force. Parallel plates and the sphere are also treated in detail.

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1 Introduction and Summary

The Casimir effect for compact regions (particularly the problem of a dieletric sphere) is still beset with difficulties, in spite of the fact that only free fields are involved [1]. The reason is that boundaries are present, which are treated by quantizing the radiation field with mode functions (see, e. g., [2]) which are adapted to the type of (classical) boundary condition, e.g. Dirichlet or Neumann. However, real boundaries consist of electrons and ions, and such boundary conditions (b.c.) are not justified except if the particles act collectively in essentially classical manner, as remarked in [2]. The ocurrence of problems is signaled by divergences occurring when Dirichlet b.c. are used [3]. In a real physical problem the latter do not occur because any material is transparent for electromagnetic radiation of sufficiently high frequency. However, "softer" conditions are usually not amenable to analytical treatment.

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An interesting idea to solve the Casimir problem was put forward by K. Dietz in a nice, but apparently not very well-known paper [4]. Following [4], consider an electromagnetic field at $T = 0$ enclosed in cavities of identical shape, but made of different materials, the latter providing natural cutoffs for the high-frequency spectrum of zero point modes. The vacuum energy is thus given by

$$E_{\rm vac} = \frac{\hbar}{2} \sum_{\alpha} \omega_\alpha C_\alpha(\Lambda) \ , \tag{1}$$

with $C_\alpha(\Lambda)$ material dependent cutoff functions dependent on a quantity Λ with dimensions of length, and which we normalize by

$$C_\alpha(\Lambda)|_{\Lambda=0} = 1 \ . \tag{2}$$

Since $E_{\rm vac}$ has dimension $(length)^{-1}$ in natural units, we expect that it may be written as an (asymptotic) series

$$E_{\rm vac} = a_0 L^3 \Lambda^{-4} + a_1 L^2 \Lambda^{-3} + a_2 L \Lambda^{-2} + a_3 \Lambda^{-1} + a_4 L^{-1} + a_5 L^{-2} \Lambda + \dots \ , \tag{3}$$

where L is a length characterizing the spatial extension of the cavity. Dietz conjectured [4] that by a theorem of Ramanujan the Λ-independent term $a_4 L^{-1}$ in (3) is *independent of the regularization* (i.e., of the set $\{C_\alpha(\Lambda)\}$ in (1)) provided (2) holds. We shall return to this remark later.

Dietz illustrated his ideas by the calculation of the Casimir effect for parallel plates computing the Λ-independent part and showing that it is the only term in (3) which contributes to the force. He then considered the vacuum energy for models where the ω_α in (1) are the eigenvalues of the Laplace-Beltrami operator with Dirichlet or Neumann b.c. on compact manifolds with boundary. He did not however, consider the Casimir effect for massless fields (e.g. the radiation field) confined in regions, where the main as yet unsolved difficulties show up [1].

In this paper we pursue Dietz's program, considering the prototypical example of massless scalar field confined in a compact region K (a compact manifold with boundary) – the modifications introduced by considering the full electromagnetic field are just of kinematical nature [1]. We show that the ω_α in (1) should be identified with the eigenvalues of the *square root* of the Laplace-Beltrami operator. This is not unexpected, because the relativistic energy is $|\vec{k}| = (\vec{k}^2)^{1/2}$, but it has important consequences for expansion (3). Consider [5, 14] the field $A(x)$ quantized in infinite space

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{\sqrt{2\omega}} \left[a(\vec{k}) e^{-ik\cdot x} + a^+(\vec{k}) e^{ik\cdot x} \right] \ ; \tag{4}$$

$$[A_-(x), A_+(y)] = \frac{1}{i} D_0^{(+)}(x-y) \ ; \tag{5}$$

$$D_0^{(+)}(x) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{-ik\cdot x} = -\frac{i}{4\pi^2} \frac{1}{(x_0 - i0)^2 - \vec{x}^2} \ . \tag{6}$$

Time evolution is generated by the Hamiltonian $H = \int d^3x H(x)$, whose density can be written in the form

$$H(x) = \frac{1}{2} : \left[\left(\frac{\partial A}{\partial x_0} \right)^2 - A \frac{\partial^2 A}{\partial x_0^2} \right] : \; . \tag{7}$$

Normal ordering is defined in momentum space. In order to go over to a geometry with *boundaries*, we should formulate it in x-space by the point-splitting technique:

$$\begin{aligned} : \left(\frac{\partial A}{\partial x_0} \right)^2 : \;&=\; \lim_{y \to x} : \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} : \\ &=\; \lim_{y \to x} \left\{ \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} - \left[\frac{\partial A_-(x)}{\partial x_0}, \frac{\partial A_+(y)}{\partial y_0} \right] \right\} \\ &=\; \lim_{y \to x} \left\{ \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} + \frac{1}{i} \frac{\partial^2}{\partial x_0^2} D_0^{(+)}(x-y) \right\} \; . \end{aligned} \tag{8}$$

Finally,

$$H(x) = \lim_{y \to x} \left\{ \frac{1}{2} \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} - \frac{1}{2} A(x) \frac{\partial^2 A(y)}{\partial y_0^2} + \frac{1}{i} \frac{\partial^2}{\partial x_0^2} D_0^{(+)}(x-y) \right\} \; . \tag{9}$$

Taking into account that real boundaries consist of electrons and ions and the field which interacts with them is quantized in *infinite space*, we consider (9) to be the Hamiltonian density describing the field both free and with boundaries. In the latter case, however, the first two terms in (9) must be defined in the adequate Fock space, i.e., the concrete representation of the field operator is dictated by the geometry. Consider a compact region K and Dirichlet b.c. $A(x) = 0$ for $\vec{x} \in \partial K$. Then $A(x)$ may be expanded as follows:

$$A(x) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[a_n u_n(\vec{x}) e^{-i\omega_n x_0} + a_n^+ u_n(\vec{x}) e^{i\omega_n x_0} \right] \; , \tag{10}$$

where u_n are normalized real eigenfunctions of the Laplacian in K, satisfying Dirichlet or Neumann b.c. (discrete spectrum):

$$-\Delta u_n(\vec{x}) = \omega_n^2 u_n(\vec{x}) \; . \tag{11}$$

The concrete Fock representation is now specified by regarding a_n^+, a_n as emission and absorption operators ($[a_n, a_m^+] = \delta_{nm}$) and defining the vacuum by

$$a_n \Omega = 0 \;\; \forall \, n \; . \tag{12}$$

We thus find in this Fock representation:

$$\begin{aligned} H(x) \;&=\; \frac{1}{2} ; \left(\frac{\partial}{\partial x_0} A(x) \right)^2 ; - \frac{1}{2} ; A(x) \frac{\partial^2}{\partial x_0^2} A(x) ; \\ &+\; \frac{1}{i} \lim_{y \to x} \frac{\partial^2}{\partial x_0^2} \left\{ D_0^{(+)}(x-y) - D_K^{(+)}(x-y) \right\} \; , \end{aligned} \tag{13a}$$

where

$$D_K^{(+)}(x_0-y_0,\vec{x},\vec{y}) = i\sum_n \frac{1}{2\omega_n}u_n(\vec{x})u_n(\vec{y})e^{-i\omega_n(x_0-y_0)} \ , \tag{13b}$$

and the semicolons in (13a) denote normal ordering with respect to the new emission and absorption operators a_n^+ and a_n. Notice that $D_0^{(+)}$ is the solution of the wave equation $\Box D_0^{(+)} = 0$ with initial conditions

$$D_0^{(+)}(+0,\vec{x}) = \frac{i}{4\pi^2}\frac{1}{\vec{x}^2+i0} \ ; \tag{14a}$$

$$\left(\partial_0 D_0^{(+)}\right)(0,\vec{x}) = \frac{1}{2}\delta(\vec{x}) \ , \tag{14b}$$

$D_0^{(+)}(+0,\vec{x})$ is the Green's function of the square root of $-\Delta$ in infinite space [5]. Similarly

$$\left(\partial_0 D_K^{(+)}\right)(x,y)\Big|_{y_0=x_0} = \frac{1}{2}\delta(\vec{x}-\vec{y}) \ ; \tag{15a}$$

$$D_K^{(+)}(+0,\vec{x},\vec{y}) = \frac{i}{2}(-\Delta_K)^{-\frac{1}{2}}(\vec{x}-\vec{y}) \ , \tag{15b}$$

where Δ_K denotes the Laplacian on K, with Dirichlet or Neumann b.c..

We now consider two types of cutoff function one of them general, satisfying (2), the other special, of type

$$C_\alpha(\Lambda) = C(\Lambda\omega_\alpha) \ , \tag{16a}$$

satisfying

$$C(0) = 1 \ . \tag{16b}$$

We shall also interested in a particular case of (16a), namely

$$C(k) = e^{-\Lambda k} \qquad k \geq 0 \ . \tag{17}$$

In terms of the special choice (17), we may, by (9), (12)-(13b), compute a regularized vacuum energy density $H_{\text{vac}}(x,\Lambda)$ in the following way:

$$\begin{aligned} H_{\text{vac}}(x,\Lambda) &= \frac{1}{2}\frac{\partial}{\partial\Lambda}\left\{\frac{1}{(2\pi)^3}\int d^3k\; e^{-i[k_0\tau-\vec{k}\cdot(\vec{x}-\vec{y})]}\Big|_{\substack{\vec{y}=\vec{x}\\ \tau=0}} C(k_0)\right. \\ &\quad - \left.\sum_n [u_n(\vec{x})]^2\, C(\omega_n)\right\} \ . \end{aligned} \tag{18}$$

As an aside, notice that (17) corresponds to ascribe a small imaginary part $-i\Lambda$ to $x_0 - y_0 = \tau$, and thus represents a "natural" choice, akin to the principal value in distribution theory [6]. For this special case (17),

$$H_{\rm vac}(\vec{x},\Lambda) = \frac{1}{2}\frac{\partial}{\partial\Lambda}\left[P(\vec{x},\vec{x};\Lambda) - P_0(\vec{x},\vec{x};\Lambda)\right] \ , \tag{19a}$$

where P, P_0 satisfy the "heat equation"

$$\left(\frac{\partial}{\partial\Lambda} - (-\Delta_{\vec{x}})^{\frac{1}{2}}\right) P(\vec{x},\vec{y};\Lambda) = 0 \ , \tag{19b}$$

with the b.c.

$$P(\vec{x},\vec{y};\Lambda) = 0 \quad \text{if } \vec{x} \ \text{ or } \vec{y} \ \in \ \partial K \ , \tag{19c}$$

in the case of Dirichlet b.c..

There exist methods to compute the asymptotic expansion (in Λ) of the quantity in brackets in (19a) [7], which solve the problem in principle, but the actual form (3), with the given coefficients, depends on the details of the discrete (eigenvalue) spectrum of $(-\Delta)^{\frac{1}{2}}$. Since the ω_α are mode functions in momentum space, we refer to the present approach ((1), (2) or (18)) as a mode method.

Let now L be a linear dimension of the compact region $K \equiv K_L$ and M a linear dimension of a region K_M of which K_L is a subset. Typically, if K_L is a cube of side L, K_M is a cube of side $M > L$ concentric with K_L, and similarly for a sphere or other manifolds. It is correct to impose the same b.c. (e.g. Dirichlet or Neumann) on K_M in order to define the outer Casimir problem [1, 9, 23]. In fact, previous work on the sphere using the Sommerfeld radiation condition was not correct, although the results are right, because it does not lead to real eigenvalues [8]. Let $H(\vec{x},\Lambda)$ be given by (1), (2) or (18), and define

$$E_{\rm vac}(L,\Lambda,M) = E_{\rm vac}^{\rm inner}(L,\Lambda) + E_{\rm vac}^{\rm outer}(L,\Lambda,M) \ , \tag{20}$$

where

$$E_{\rm vac}^{\rm inner}(L,\Lambda) \equiv \int_{K_L} d\vec{x} \ H(\vec{x},\Lambda) \ , \tag{21}$$

and

$$E_{\rm vac}^{\rm outer}(L,\Lambda,M) = \int_{K_M \backslash K_L} d\vec{x} \ \tilde{H}(\vec{x},\Lambda) \ . \tag{22}$$

As previously remarked, if Dirichlet b.c. are imposed on K_L, $\tilde{H}$ is the density (18) with Dirichlet b.c. imposed on K_L and K_M, for definiteness. Analogous definitions hold for other b.c. (e.g Neumann or mixed).

If (1), (2) is adopted, the second sum in (20) refers, then, to the modes ω_n corresponding to the solution of (11) in $K_M \backslash K_L$, with the above-mentioned b.c.. Suppose that both $E_{\rm vac}^{\rm inner}(L,\Lambda)$ and $E_{\rm vac}^{\rm outer}(L,\Lambda,M)$ have asymptotic series (3), and let $E_{\rm vac}^{\rm inner}(L)(\equiv a_4^{\rm inner}/L)$ and $E_{\rm vac}^{\rm outer}(L,M)$ be the corresponding Λ-independent terms.

Then the Casimir pressure $p_C(L)$ (a measurable quantity) is defined by the thermodynamic formulae (zero absolute temperature):

$$p_C(L) = p_C^{\text{inner}}(L) - p_C^{\text{outer}}(L) \ , \tag{23a}$$

where the relative minus sign takes into account that p_C^{outer} points inward K_L whereas p_C^{inner} is directed outward, and

$$p_C^{\text{inner}}(L) = -\frac{\partial E_{\text{vac}}^{\text{inner}}(L)}{\partial V_{\text{inner}}(L)} \ ; \tag{23b}$$

$$p_C^{\text{outer}}(L) = -\lim_{M\to\infty} \frac{\partial E_{\text{vac}}^{\text{outer}}(L,M)}{\partial V_{\text{outer}}(L,M)} \ . \tag{23c}$$

For the cube of side L, $V_{\text{inner}}(L) = L^3$ and $V_{\text{outer}}(L,M) = M^3 - L^3$, if the outer cube is concentric with the inner cube and of side M. Other cases are of course analogous; what is important here is that M is *fixed*, only L varies. The limit (23c) is expected to exist on the basis of general results on the thermodynamic limit [10]; this will be verified explicitly for the cube. For the sphere, is $E_{\text{vac}}^{\text{inner}}(L,\Lambda) + E_{\text{vac}}^{\text{outer}}(L,\Lambda,M)$ which have an asymptotic series (3) as $M \to \infty$, so that for the sphere of radius a we have (see the conclusion):

$$p_C(a) = -\frac{1}{4\pi a^2}\frac{\partial}{\partial a}\frac{a_4^{\text{sphere}}}{a} = \frac{a_4^{\text{sphere}}}{4\pi a^4} \ . \tag{24}$$

It is, of course, highly desirable that the CE be independent of the cutoff function C in (1) or (16a) provided it satisfies (2) or (16b). As remarked in [4], a necessary condition for this regularization independence (RI) to hold is that (3) contain no logarithmic terms, because, otherwise, the "Λ-independent term" is obviously ill-defined. For the cube we shall see that there are no logarithmic terms in (3) in either the inner or the outer integrals in (20), but such is not the case for the sphere, where only the full expression is free of logarithms. A full proof of RI is given in section 2 for parallel plates. The same method of proof, which is due to Ramanujan, should work in the other examples we treat, and basically, to prove RI for the Casimir effect in "well-behaved" compact manifolds (e.g. with boundary containing at most a finite number of singular points), following refs. [11], but we do not attempt that here‡. We content ourselves with a few important remarks:

a) the Λ-independent term in (3) should coincide with the "Ramanujan sum" of a divergent series of positive terms, such as (1), with $C_\alpha(\Lambda) \equiv 1$ see [12] (p.

‡In fact, preliminary work generalizing refs. [11] (in order to include the corresponding to the second term in the r.h.s. of (45) – which is crucial in the proof of RI) indicates that the same proof holds in higher dimensions.

318 ff.) and section 1. According to this concept, for instance

$$1 + 1 + \cdots + 1 + \cdots = -\frac{1}{2}\ (\Re, 0)\ ;$$

$$1 + 2 + 3 + \cdots = -\frac{1}{12}\ (\Re, 0)\ ,$$

taking the origin as reference point (see [12], 13.10.11);

b) the present definition of the CE employs only well-defined mathematical concepts. In particular, the limit $\Lambda \to 0$ is never taken. In fact, (3) shows that, in general, it does not exist (an exception is the Casimir effect for parallel plates with periodic b.c., see [5] and section 2). The reason for this is that we do not know how to treat the surface properly in microscopic terms, a formidable problem. Nevertheless, RI justifies the definition physically: the Λ-independent term reflects the field-theoretic structure of the vacuum state which is independent of the cavity materials.

In the context of *b*), the reader may well ask why all this fuss about RI. Isn't it physically obvious? Yes, but present theory does not substantiate this, as emphasized by Candelas in [13]. Indeed for a sphere of radius R, with Dirichlet b.c. one obtains [14]

$$\frac{1}{2}(-\Delta)^{-\frac{1}{2}}(\vec{x}, \vec{y}) = \frac{1}{4\pi^2 \left|\vec{x} - \vec{y}\right|^2 + i0} - \frac{R^2}{4\pi^2 \left|\vec{y}\right|^2 \left|\vec{x} - \vec{y}_*\right|^2 + i0}\ , \tag{25}$$

by the method of images (lest some doubts arise on the derivation of (25), we have clarified some points in Appendix A). Above,

$$\vec{y}_* = \frac{\vec{y}R^2}{\left|\vec{y}\right|^2}\ , \tag{26}$$

is the image point. The singular free space term cancels in the formula for the energy density $u_R(\vec{x})$ based on (12) and derived in [14], whereby one obtains (13) of ref. [14]:

$$u_R(\vec{x}) = \frac{R^2 \left|\vec{x}\right|^2}{2\pi^2(\left|\vec{x}\right|^2 - R^2)^4}\ . \tag{27}$$

Defining (as in (20))

$$E_{\text{vac}}^{\delta_1,\delta_2} = \int_{K_R^{\delta_1}} d\vec{x}\ u_R(\vec{x}) + \int_{\Omega \backslash K_R^{\delta_2}} d\vec{x}\ u_R(\vec{x})\ , \tag{28}$$

where $K_R^{\delta_1} = \{\vec{x} | \left|\vec{x}\right| \le R - \delta_1\}$, and $K_R^{\delta_2} = \{\vec{x} | \left|\vec{x}\right| \le R + \delta_2\}$, one obtains terms $R^2\delta_1^{-3}$, $R\delta_1^{-2}$, δ_1^{-1}, $\log \frac{\delta_1}{R}$ in the first integral of (28), and analogous ones for the second integral involving δ_2. Choosing $\delta_1 \neq \delta_2$, one may achieve [14] that all singular terms cancel in the force, and the finite part

$$E_{\rm reg}(R) = -\frac{1}{96\pi R} , \tag{29}$$

survives [14].

We have repeated these results and arguments of [14] to display the marked differences between the configuration ($\vec{x}$-)space prescription and the mode approach: in the $\vec{x}$-space method there is no RI (in general there are logarithmic terms in the sum (28)), and, furthermore, (29) does not agree with the mode result even qualitatively (see section 4; for the sphere a repulsive force is obtained). In the case of penetrable sphere or dieletric ball, it was shown [15] that RI holds in the dilute case only. Incidentally, there is an important quantum field theoretic problem where RI does not hold: dynamical mass generation, e.g., in QED_3 [16]. In the latter problem there is a consistent approach within the causal theory [17].

Section 2 entails a complete proof of RI for parallel plates (a "limit" of a compact region), as well as the explanation of the "theory without cutoffs" for the case of periodic boundary conditions in ref. [5]. In section 3 we illustrate the present theory by computing the CE for a cube and in section 4 we calculate the CE for a sphere. In section 5 we discuss the results and briefly present some open problems. Appendix A clarifies briefly some points in the ($\vec{x}$-space) calculation of [14] for the sphere. Appendix B brings together some useful asymptotic formulae for the modified Bessel functions.

2 Parallel Plates

We consider the problem of parallel plates, with distance d along the z-axis; take the positions of the plates at $z = 0$ and $z = d$, and adopt the form (16a) in (18), (20) with Dirichlet b.c. (Neumann b.c. yield the same results). The *inner* Casimir problem corresponds to the region $K_L = K_d = \{\vec{x} \in \mathbb{R}^2 \times [0, d]\}$, and the *outer* one to the region $K_R \backslash K_L = \{\vec{x} \in \mathbb{R}^2 \times [d, d + R]\} \cup \{\vec{x} \in \mathbb{R}^2 \times [-R, 0]\}$. The eigenfunctions associated to the inner problem are

$$u_n^{\rm inner}(k_x, k_y) = \frac{1}{2\pi}\sqrt{\frac{2}{d}} \sin\left(\frac{n\pi}{d} z\right) e^{i(k_x x + k_y y)} \qquad n = 1, 2, 3 \cdots , \tag{30}$$

corresponding to the eigenvalues of $(-\Delta)^{\frac{1}{2}}$ given by

$$\omega_{n,k_x,k_y}^{\rm inner} = \sqrt{\left(\frac{n\pi}{d}\right)^2 + k_x^2 + k_y^2} , \tag{31}$$

in (11). The outer eigenfunctions are

$$u_n^{\rm outer,1}(k_x, k_y) = \frac{1}{2\pi}\sqrt{\frac{2}{R}} \sin\left(\frac{n\pi}{R}(z - d)\right) e^{i(k_x x + k_y y)} ;$$

$$u_n^{\text{outer},2}(k_x,k_y) = \frac{1}{2\pi}\sqrt{\frac{2}{R}}\sin\left(\frac{n\pi}{R}z\right)e^{i(k_x x + k_y y)} \ , \tag{32}$$

with eigenvalues

$$\omega^{\text{outer}}_{n,k_x,k_y} = \sqrt{\left(\frac{n\pi}{R}\right)^2 + k_x^2 + k_y^2} \ . \tag{33}$$

We first adopt the choice (17). Introducing polar coordinates in the x-y plane, we calculate the first (inner) sum in (20) (we do *not* integrate along $(x,y)\in\mathbb{R}^2$, which would yield $+\infty$). The proper way to do this is to limit the (x-y)-plane integration to a finite region with area A, and then take the limit for $\mathcal{E}=\frac{E}{A}$ (this procedure yields the same results presented here and we omit it for brevity):

$$\begin{aligned}\mathcal{E}^{\text{inner}}_{\text{vac}}(\Lambda,d) &= \frac{1}{2(2\pi)^2}\left\{-2d\int_0^\infty dk\ k^3\ e^{-\Lambda k}\right.\\ &+ \left. 2\pi\sum_{n=1}^{\infty}\int_0^\infty dk\ k\ e^{-\Lambda\sqrt{\left(\frac{n\pi}{d}\right)^2+k^2}}\ \sqrt{\left(\frac{n\pi}{d}\right)^2+k^2}\right\} \ .\end{aligned} \tag{34}$$

Performing the change of variable $k'_n = \sqrt{\left(\frac{n\pi}{d}\right)^2+k^2}$ in the second integral in the r.h.s. of (34) we obtain

$$\begin{aligned}\mathcal{E}^{\text{inner}}_{\text{vac}}(\Lambda,d) &= \frac{1}{2(2\pi)^2}\left\{-2d\int_0^\infty dk\ k^3\ e^{-\Lambda k} + 2\pi\sum_{n=1}^{\infty}\int_{\frac{n\pi}{d}}^\infty dk'_n\ k'^2_n\ e^{-\Lambda k'_n}\right\}\\ &= \frac{d}{(2\pi)^2}\left\{-6\Lambda^{-4} + \frac{\partial^2}{\partial\Lambda^2}\left[\frac{1}{\Lambda^2}\frac{\frac{\Lambda\pi}{d}}{e^{\frac{\Lambda\pi}{d}}-1}\right]\right\} \ ,\end{aligned} \tag{35}$$

we now use the expansion ([12], p. 320) in (35)

$$\frac{t}{e^t-1} = 1 - \frac{1}{2}t + \sum_{k=1}^{\infty}(-1)^{k-1}B_k\frac{t^{2k}}{(2k)!} \ , \tag{36}$$

obtaining ($B_2=\frac{1}{30}$):

$$\mathcal{E}^{\text{inner}}_{\text{vac}}(\Lambda,d) = -\frac{1}{4\pi\Lambda^3} - \frac{1}{2}\,\frac{\pi^2}{720 d^3} + \mathcal{O}(\Lambda) \ , \tag{37}$$

and thus

$$\mathcal{E}^{\text{inner}}_{\text{Casimir}} = -\frac{1}{2}\,\frac{\pi^2}{720 d^3} \ . \tag{38}$$

Two remarks are in order. The surface term $-\frac{1}{4\pi\Lambda^3}$ in (37) are absent for periodic b.c., because the latter allow the term $n=0$ in (34) which exactly cancels it. This explains the result of [5]. The external CE is *zero* due to (32), (33) because, for the outer problem, d in (37) is replaced by R, and thus in the limit $R \to \infty$

$$\mathcal{E}_{\text{Casimir}}^{\text{outer}} = 0 \; . \tag{39}$$

Finally,

$$\mathcal{E}_{\text{Casimir}} = -\frac{1}{2} \, \frac{\pi^2}{720 d^3} \; . \tag{40}$$

The above energy is one half of the result for the electromagnetic field, due to the summation over the two polarization states in the latter. Notice also that, in natural units, $\mathcal{E}$ is of order $(length)^{-3}$.

An amusing aspect of the present derivation is that it seems to depend on the choice (17), i.e., of an exponential cutoff in (34) and (35), which, due to (36), leads to (37). Consider now a general cutoff function (16a). Omitting the volume term in (34), we may write

$$\mathcal{E}_{\text{vac}}^{\text{inner}}(\Lambda, d) = \lim_{n\to\infty} \frac{1}{8\pi} \sum_{m=1}^{n} g(m) \; , \tag{41}$$

where

$$\begin{aligned} g(m) &= \int_0^\infty du \, \sqrt{u + \left(\frac{m\pi}{d}\right)^2} \, C\left(\Lambda \sqrt{u + \left(\frac{m\pi}{d}\right)^2}\right) \\ &= \int_{\left(\frac{m\pi}{d}\right)^2}^\infty du \, \sqrt{u} \, C(\Lambda \sqrt{u}) \; . \end{aligned} \tag{42}$$

It is of interest to compute

$$\frac{d}{\pi} g^{(1)}(m) = -2 \left(\frac{m\pi}{d}\right)^2 C\left(\Lambda \frac{m\pi}{d}\right) \; ; \tag{43a}$$

$$\frac{d}{\pi} g^{(2)}(m) = -4 \frac{\pi}{d} \left(\frac{m\pi}{d}\right) C\left(\Lambda \frac{m\pi}{d}\right) - 2 \left(\frac{m\pi}{d}\right)^2 \left(\frac{\Lambda\pi}{d}\right) C^{(1)}\left(\Lambda \frac{m\pi}{d}\right) \; ; \tag{43b}$$

$$\begin{aligned} \frac{d}{\pi} g^{(3)}(m) &= -4 \left(\frac{\pi}{d}\right)^2 C\left(\Lambda \frac{m\pi}{d}\right) - 8 \frac{\pi}{d} \left(\frac{m\pi}{d}\right) \left(\frac{\Lambda\pi}{d}\right) C^{(1)}\left(\Lambda \frac{m\pi}{d}\right) \\ &- 2 \left(\frac{m\pi}{d}\right)^2 \left(\frac{\Lambda\pi}{d}\right)^2 C^{(2)}\left(\Lambda \frac{m\pi}{d}\right) \; ; \end{aligned} \tag{43c}$$

$$\begin{aligned} \frac{d}{\pi} g^{(4)}(m) &= -12 \left(\frac{\pi}{d}\right)^2 \left(\frac{\Lambda\pi}{d}\right) C^{(1)}\left(\Lambda \frac{m\pi}{d}\right) - 12 \frac{\pi}{d} \left(\frac{m\pi}{d}\right) \left(\frac{\Lambda\pi}{d}\right)^2 C^{(2)}\left(\Lambda \frac{m\pi}{d}\right) \\ &- 2 \left(\frac{m\pi}{d}\right)^2 \left(\frac{\Lambda\pi}{d}\right)^3 C^{(3)}\left(\Lambda \frac{m\pi}{d}\right) \; . \end{aligned} \tag{43d}$$

By [12] (p. 326), under the following conditions (49) and (50) on C:

$$\sum_{m=1}^{n} g(m) - \frac{2d}{\pi} \int_0^\infty dk\ k^3\ C(\Lambda k) + \frac{1}{2} g(0) \quad \underset{n\to\infty}{\longrightarrow} \Sigma_k\ , \tag{44}$$

where

$$\Sigma_k = -S_k(0) - \frac{1}{(2k+2)!} \int_0^\infty \psi_{2k+2}(t) g^{(2k+2)}(t) dt\ , \tag{45}$$

and

$$\psi_k(x) = \phi_k(x) \bmod 1 \qquad \text{(i.e., equal to } \phi_k(x) \text{ for } 0 \le x < 1 \text{ with period 1)}\ , \tag{46}$$

and ϕ_k are defined by

$$t \frac{e^{xt} - 1}{e^t - 1} = \sum_{n=1}^{\infty} \phi_k(x) \frac{t^n}{n!}\ , \tag{47}$$

and

$$S_k(0) = \sum_{r=1}^{k} (-1)^{r-1} \frac{B_r}{(2r)!} g^{(2r-1)}(0)\ . \tag{48}$$

We changed the notation of [12]: the C_k on pg. 326 corresponds to our Σ_k. Notice that the integral in (44) is the first (volume) term on the r.h.s. of (34) and $\frac{1}{2}g(0)$ contributes only to the Λ-dependent terms in the asymptotic series. We assume that C satisfies, besides (16b), the conditions: C is infinitely differentiable and its derivatives $C^{(k)}$ ($C^{(0)} \equiv C$) satisfy

$$\int^\infty C^{(k)}(x) dx < \infty\ ; \tag{49}$$

$$C^{(k)}(x) \underset{x\to\infty}{\longrightarrow} 0\ , \tag{50}$$

for all $k \ge 0$. It follows then (see [12], pp 326 ff.) that Σ_k *is independent of* k, *for* $k \ge 1$. This is related to the RI referred to in the main text. Σ_k ($k \ge 1$) is referred to as the $(\Re, 0)$ sum of the (divergent) series $\sum_{m=1}^\infty g(m)$, where $\Re$ refers to Ramanujan and 0 to the reference point (the origin in our case). Usually (see, e.g., [18], p. 138), the result is presented informally without the important last term in (45), and assuming that C satisfies $C^{(k)}(0) = 0$ for all $k \ge 1$, besides (16b), which is not satisfied by the special choice (17). We now prove that, under the conditions (16b), (49) and (50), the same result (40) is obtained for the CE, which is the precise expression of RI in this case.

Proof. Since Σ_k is independent of k, for $k \ge 1$, choose $k = 2$. By (45)-(48),

$$\Sigma_2 = -\frac{B_1}{2} g^{(1)}(0) + \frac{B_2}{24} g^{(3)}(0) - \frac{1}{6!} \int_0^\infty \psi_6(t) g^{(6)}(t) dt\ . \tag{51}$$

Putting (43a), (43c) and (16b) into (51), we find

$$\Sigma_2 = -\frac{B_2}{6}\left(\frac{\pi}{d}\right)^3 + \mathcal{O}(\Lambda^2)\ , \tag{52}$$

which leads to (40) by (41). The term $\mathcal{O}(\Lambda^2)$ in (52) comes from derivating $g^{(4)}$ in (43d) further twice, and making the change of variable $t' = \frac{\Lambda\pi}{d}t$ in the integral in (51), taking into account that ψ_k is $\mathcal{O}(1)$.

What if we choose $k = 1$? By (48) and (43a), $S_1(0) = 0$, but, in (45), we still have the second term

$$\Sigma_1 = -\frac{1}{24}\int_0^\infty \psi_4(t) g^{(4)}(t) dt\ . \tag{53}$$

We use the recurrence ([12], 13.2.13)

$$\psi^{(1)}_{2m+1} = (2m+1)\left\{\psi_{2m} + (-1)^{m-1}B_m\right\}\ , \tag{54}$$

with $m = 2$, obtaining

$$\psi_4 - B_2 = \frac{1}{5}\psi_5^{(1)}\ , \tag{55}$$

which we insert in (53), getting

$$\Sigma_1 = -\frac{1}{24}\int_0^\infty \frac{\psi_5^{(1)}(t)}{5} g^{(4)}(t) dt - \frac{1}{24}B_2\int_0^\infty g^{(4)}(t) dt\ . \tag{56}$$

Integration by parts in the first term on the r.h.s. of (56) and use of (43c) in the second term yield (using $\psi_n(0) = 0$)

$$\Sigma_1 = \frac{1}{120}\int_0^\infty \psi_5(t) g^{(5)}(t) dt + \frac{B_2}{24} g^{(3)}(0)\ . \tag{57}$$

A further integration by parts using the recurrence ([12], 13.2.13)

$$\psi^{(1)}_{2m} = 2m\psi_{2m-1}\ , \tag{58}$$

brings (57) to the form (51). We have thus proved

$$\Sigma_k = -\frac{B_2}{6}\left(\frac{\pi}{d}\right)^3 + \mathcal{O}(\Lambda^2)\ , \tag{59}$$

for all $k \geq 1$ (the present argument is easily generalized). Thus, for parallel plates and Dirichlet b.c. the Λ-independent term in the asymptotic series (3) is regularization independent and is the $(\Re, 0)$ sum of the divergent series (41). Neumann b.c. yield the same result.

3 The Cube

Consider now the explicit case of a cube K of side L, with Dirichlet b.c. (Neumann b.c. may be handled analogously). The normalized eigenfunctions and eigenvalues of $(-\Delta)^{\frac{1}{2}}$ are

$$\begin{aligned}
&u_{n_1n_2n_3}(\vec{x}) = \left(\frac{2}{L}\right)^{\frac{3}{2}} \sin\left(\frac{n_1\pi x_1}{L}\right) \sin\left(\frac{n_2\pi x_2}{L}\right) \sin\left(\frac{n_3\pi x_3}{L}\right) \ ; \\
&n_1 = 1, 2, \cdots \ ; \quad n_2 = 1, 2, \cdots \ ; \quad n_3 = 1, 2, \cdots \ ; \\
&(-\Delta)^{\frac{1}{2}} u_{n_1n_2n_3}(\vec{x}) = \frac{\pi}{L}|\vec{n}| u_{n_1n_2n_3}(\vec{x}) \ ; \\
&|\vec{n}| = (n_1^2 + n_2^2 + n_3^2)^{\frac{1}{2}} \ .
\end{aligned} \tag{60}$$

We consider

$$E_{\text{vac}}(\Lambda) = \int_K d\vec{x} \ H(\vec{x}, \Lambda) \ . \tag{61}$$

By (13a),

$$E_{\text{vac}}(\Lambda) = \frac{1}{2}\frac{\partial}{\partial\Lambda}\left\{\frac{L^3}{(2\pi)^3}\int d^3k \ e^{-\Lambda|\vec{k}|} - \sum_{\vec{n}} e^{-\Lambda\omega_{\vec{n}}}\right\} \ . \tag{62}$$

By (60), $\omega_{\vec{n}} = \frac{\pi}{L}|\vec{n}|$ and hence

$$E_{\text{vac}}(\Lambda) = -\frac{3}{2\pi^2}L^3\Lambda^{-4} - \frac{1}{2}\frac{\partial}{\partial\Lambda}\left\{\frac{1}{8}\left[\sum_{\vec{n}\in\mathbb{Z}^3} e^{-a|\vec{n}|} - 3\sum_{\vec{n}\in\mathbb{Z}^2} e^{-a|\vec{n}|} + 3\sum_{n\in\mathbb{Z}} e^{-a|\vec{n}|} - 1\right]\right\} \ , \tag{63}$$

where

$$a = \frac{\pi}{L}\Lambda \ . \tag{64}$$

The last sums in (63) are due the fact that, because of (60), the planes $n_1 = 0$, $n_2 = 0$, and $n_3 = 0$ have to be excluded from the sum over $\mathbb{Z}^3$ because they lead to eigenfunctions which are zero. For the same reason the axes $n_1 = n_2 = 0$, $n_1 = n_3 = 0$, $n_2 = n_3 = 0$ and the origin be excluded. Exclusion of the three planes (the term $-3\sum_{\vec{n}\in\mathbb{Z}^2} e^{-a|\vec{n}|}$ in (63)) corresponds to excluded each axis twice instead of only once. The third term compensates for this, while the last one excludes the origin.

A method of calculation of the lattice sums in (63) is through the Poisson summation formula

$$\sum_{\vec{n}\in\mathbb{Z}^3} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^3} C_{\vec{m}} \ , \tag{65}$$

where $C_{\vec{m}}$ are the Fourier coefficients of f:

$$C_{\vec{m}} = \int e^{-2\pi i \vec{m}\cdot\vec{x}} f(\vec{x}) d\vec{x} \ . \tag{66}$$

See also ref. [22]. Applying (65) to (63), we find

$$\begin{aligned} E_{\text{vac}}(\Lambda) &= -\frac{3}{2\pi^2} L^3 \Lambda^{-4} - \frac{1}{16} \frac{\partial}{\partial \Lambda} \Bigg\{ \sum_{\substack{\vec{m}\in\mathbb{Z}^3 \\ \vec{m}\neq\vec{0}}} 2\pi \int_{-1}^{1} du \int_0^\infty dr\ r^2 e^{-2\pi i |\vec{m}| r u} e^{-ar} \\ &- 3 \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} 2\pi \int_0^\infty dr\ r e^{-ar} J_0(2\pi|\vec{m}|r) + \frac{8}{\pi^2} L^3 \Lambda^{-3} - \frac{6}{\pi} L^2 \Lambda^{-2} \\ &+ 3 \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \int_{-\infty}^{\infty} dx e^{-2\pi i m x} e^{-a|x|} + \frac{6}{\pi} L \Lambda^{-1} - 1 \Bigg\} \ . \end{aligned} \tag{67}$$

We thus find

$$\begin{aligned} E_{\text{vac}}(\Lambda) &= -\frac{3}{2\pi^2} L^3 \Lambda^{-4} + \frac{3}{2\pi^2} L^3 \Lambda^{-4} - \frac{3}{4\pi} L^2 \Lambda^{-3} + \frac{3}{8\pi} L \Lambda^{-2} \\ &- \frac{\pi^2}{2L} \sum_{\substack{\vec{m}\in\mathbb{Z}^3 \\ \vec{m}\neq\vec{0}}} \left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2 \right]^{-2} + \frac{2\pi^4}{L} \left(\frac{\Lambda}{L}\right)^2 \sum_{\substack{\vec{m}\in\mathbb{Z}^3 \\ \vec{m}\neq\vec{0}}} \left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2 \right]^{-3} \\ &+ \frac{3\pi^2}{8L} \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} \left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2 \right]^{-\frac{3}{2}} - \frac{9\pi^4}{8L} \left(\frac{\Lambda}{L}\right)^2 \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} \left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2 \right]^{-\frac{5}{2}} \\ &- \frac{3\pi}{8L} \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2 m^2 \right]^{-1} + \frac{3\pi^3}{4L} \left(\frac{\Lambda}{L}\right)^2 \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2 m^2 \right]^{-2} \ . \end{aligned} \tag{68}$$

We now expand the sums $\sum_{\vec{m}\neq\vec{0}}$ in (68) in the following way:

$$\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2 \right]^{-s} = \left(4\pi^2|\vec{m}|^2\right)^{-s} \left(1 - \frac{s\Lambda^2}{4L^2|\vec{m}|^2} + \cdots \right) \ . \tag{69}$$

The unit term in (69) yields a contribution of type $a_4 L^{-1}$ in (3), the remaining terms provide the rest of the asymptotic series in (3) consisting of positive powers of Λ.

We thus find

$$a_4 = -\frac{1}{32\pi^2} \sum_{\substack{\vec{m} \in \mathbb{Z}^3 \\ \vec{m} \neq \vec{0}}} |\vec{m}|^{-4} + \frac{3}{64\pi} \sum_{\substack{\vec{m} \in \mathbb{Z}^2 \\ \vec{m} \neq \vec{0}}} |\vec{m}|^{-3} - \frac{3}{32\pi} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} m^{-2} . \tag{70}$$

The last sum above is nothing but $2\zeta(2)$, where ζ stands for the Riemann zeta function, and the second one may be rewritten as two independent sums by means of (see, e.g., [19, 20])

$$\sum_{\substack{\vec{m} \in \mathbb{Z}^2 \\ \vec{m} \neq \vec{0}}} |\vec{m}|^{-s} = 4\zeta\left(\frac{s}{2}\right) \beta\left(\frac{s}{2}\right) , \tag{71}$$

where

$$\beta(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^s} . \tag{72}$$

The first sum in (70) was calculated by Lukosz [21] who obtained

$$\sum_{\substack{\vec{m} \in \mathbb{Z}^3 \\ \vec{m} \neq \vec{0}}} |\vec{m}|^{-4} = 16.53231596\ldots . \tag{73}$$

Then, substituting (73) and the explicit values for $\zeta(2)$, $\zeta(\frac{3}{2})$ and $\beta(\frac{3}{2})$ in (70) we obtain

$$a_4 = -0.01573\ldots , \tag{74}$$

which is in accordance with the result obtained numerically in ref. [19] (in fact we have obtained a_4 to a higher accuracy than shown). In addition, from (23b), the inner pressure is

$$p_C^{\text{inner}}(L) = \frac{a_4}{3L^4} . \tag{75}$$

It is of great interest to consider also the outer problem for the cube, a difficult problem which has never been solved. We will consider the cube K_L of side L concentric with a cube K_M, of side M, from which K_L is a subset ($M > L$ and M eventually goes to infinity at the end of calculation) and impose Dirichlet b.c. on K_L as well as K_M (see section 1). Unfortunately, the solution of the external Casimir problem for the cube with Dirichlet b.c. cannot be constructed out of the functions of the form (60), because the continuity conditions on several planes cannot be satisfied simultaneously. However, the form of solutions (60), which are naturally adapted to the internal geometry of the cube, suggest splitting the region $K_M \backslash K_L$ into 26 subregions bounded by the planes containing the faces of the cube. We may require the $u_n(\vec{x})$ to vanish on the boundaries of these subregions, including the original requirement of vanishing on the faces of the internal and external cubes. If we do so, the resulting problem is explicitly solvable in terms of the set (60). Of course, this procedure introduces additional stresses in the region $K_M \backslash K_L$. We shall comment on these restrictions in the conclusion.

Then we have that the 26 subregions which compose $K_M \backslash K_L$ are of three topologically distinct kinds (with both cubes centered in the origin): 1) a rectangular box with two sides L and one $\frac{M-L}{2}$ – with multiplicity 6; 2) a rectangular box with two sides $\frac{M-L}{2}$ and one L (with multiplicity 12) – the contribution of the edges; 3) a cube of sides $\frac{M-L}{2}$ (with multiplicity 8) – the contribution of the corners. The Casimir energy of each of these regions can be obtained along the same lines of the calculation above outlined for the inner cube (see [22]). Then we obtain that the regions of kind 3) do not contribute, i.e., there are no contributions of the corners. The total contribution of the regions of kind 1) is

$$
\begin{aligned}
E_1(L,M) &= -\frac{3L^2(M-L)}{32\pi^2} \sum_{\substack{\vec{m}\in\mathbb{Z}^3 \\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2L^2 + m_3^2\frac{(M-L)^2}{4}\right]^2} \\
&+ \frac{3L(M-L)}{32\pi} \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4}\right]^{\frac{3}{2}}} \\
&+ \frac{3}{32\pi L} \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} \frac{1}{[m_1^2 + m_2^2]^{\frac{3}{2}}} - \frac{\pi}{16}\left(\frac{2}{L} + \frac{2}{M-L}\right) ,
\end{aligned}
\tag{76}
$$

and for the regions of kind 2) the total contribution is

$$
\begin{aligned}
E_2(L,M) &= -\frac{3L(M-L)^2}{32\pi^2} \sum_{\substack{\vec{m}\in\mathbb{Z}^3 \\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4} + m_3^2\frac{(M-L)^2}{4}\right]^2} \\
&+ \frac{3L(M-L)}{16\pi} \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4}\right]^{\frac{3}{2}}} \\
&+ \frac{3}{8\pi(M-L)} \sum_{\substack{\vec{m}\in\mathbb{Z}^2 \\ \vec{m}\neq\vec{0}}} \frac{1}{[m_1^2 + m_2^2]^{\frac{3}{2}}} - \frac{\pi}{8}\left(\frac{1}{L} + \frac{4}{M-L}\right) .
\end{aligned}
\tag{77}
$$

Of course these energies are not finite in the limit $M \to \infty$, but even yet its contributions for the pressure are finite and well defined. For the kind 1) regions we have from equation (23c) that

$$
p_1(L) \equiv \lim_{M\to\infty} p_1(L,M) , \tag{78}
$$

where (remember that M is fixed)

$$
p_1(L,M) = -\frac{\partial E_1(L,M)}{\partial V_1} = -\frac{2}{2ML - 3L^2}\frac{\partial E_1(L,M)}{\partial L} . \tag{79}
$$

It must be noted that in the equations (78)-(79) the order in which we take the limit and the derivative must be carefully observed, otherwise we may obtain a wrong result. Then, we obtain

$$
\begin{aligned}
p_1(L,M) &= \frac{3}{16\pi^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}}\frac{1}{\left[m_1^2L^2+m_2^2L^2+m_3^2\frac{(M-L)^2}{4}\right]^2}\\
&- \frac{3}{4\pi^2}\frac{L^2(M-L)}{2LM-3L^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}}\frac{m_1^2L+m_2^2L+m_3^2\frac{(L-M)}{4}}{\left[m_1^2L^2+m_2^2L^2+m_3^2\frac{(M-L)^2}{4}\right]^3}\\
&- \frac{3}{16\pi}\frac{M-2L}{2ML-3L^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\frac{1}{\left[m_1^2L^2+m_2^2\frac{(M-L)^2}{4}\right]^{\frac{3}{2}}}\\
&+ \frac{9}{16\pi}\frac{L(M-L)}{2ML-3L^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\frac{m_1^2L+m_2^2\frac{(L-M)}{4}}{\left[m_1^2L^2+m_2^2\frac{(M-L)^2}{4}\right]^{\frac{3}{2}}}\\
&+ \frac{3}{16\pi}\frac{1}{L^2(2ML-3L^2)}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\frac{1}{[m_1^2+m_2^2]^{\frac{3}{2}}}\\
&+ \frac{\pi}{8}\frac{1}{2ML-3L^2}\left(\frac{2}{(M-L)^2}-\frac{2}{L^2}\right) \qquad (80)
\end{aligned}
$$

In the limit $M\to\infty$ we note that in the above sums the terms in which the summation index multiplying $\frac{(M-L)^2}{4}$ is different of zero do not contribute because, when $M\gg L$, we have, for example,

$$
\begin{aligned}
\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ m_3\neq 0}}\frac{1}{\left[m_1^2L^2+m_2^2L^2+m_3^2\frac{(M-L)^2}{4}\right]^2} &\leq \left(\frac{2}{M-L}\right)^{\frac{1}{2}}\\
&\times \sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ m_3\neq 0}}\frac{1}{\left[m_1^2L^2+m_2^2L^2+m_3^2\frac{(M-L)^2}{4}\right]^{\frac{7}{4}}}\\
&\leq \left(\frac{2}{L^7(M-L)}\right)^{\frac{1}{2}}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ m_3\neq 0}}\frac{1}{[m_1^2+m_2^2+m_3^2]^{\frac{7}{4}}}\\
&\to 0\,, \qquad \text{when} \;\; M\to\infty\,,
\end{aligned}
$$

and we may easily check that the same holds for the other sums of the same kind in (80). Then we obtain

$$
p_1(L)=\frac{1}{L^4}\left\{-\frac{3}{16\pi^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ m_3\neq 0}}|\vec{m}|^{-4}+\frac{3}{16\pi}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}|m|^{-3}\right\}\,, \qquad (81)
$$

or, by using (71),

$$p_1(L) = \frac{1}{L^4}\left\{\frac{3}{8\pi}\zeta(3) - \frac{3}{4\pi^2}\zeta(2)\beta(2)\right\} . \tag{82}$$

In the same way for the regions of kind 2) – the ridge contribution – we have

$$p_2(L) \equiv \lim_{M\to\infty} p_2(L, M) , \tag{83}$$

where

$$p_2(L, M) = -\frac{\partial E_2(L, M)}{\partial V_2} = -\frac{4}{(M-L)(M-3L)}\frac{\partial E_2(L, M)}{\partial L} , \tag{84}$$

and following along the same steps above we obtain

$$p_2(L) = -\frac{9}{4\pi^2}\zeta(4)\frac{1}{L^4} . \tag{85}$$

Then, the outer pressure is given by the sum of the contributions of these two kind of regions:

$$p_C^{\mathrm{outer}}(L) = p_1(L) + p_2(L) , \tag{86}$$

and it results

$$p_C^{\mathrm{outer}} = -\frac{0.21775\cdots}{L^4} , \tag{87}$$

so that the Casimir pressure is repulsive:

$$p_C(L) = p_C^{\mathrm{inner}} - p_C^{\mathrm{outer}} \simeq \frac{0.21251}{L^4} . \tag{88}$$

It is important to note that the edges contribution ($p_2(L)$), which defines the effect of curvature, dominates in (86)-(88), *determining the sign* of the Casimir effect. We shall return to this point in the conclusion.

4 The Sphere

The Casimir effect for b.c. on a sphere was first considered in the classic paper by Boyer [23] and since it has been considered from diverse viewpoints: source theory [24], multiple scattering [25], dimensional dependence of the effect [26, 27] as well as an improved mode summation method [8, 9] (for more detailed reference list see [1] and [28]).

Here we will to reconsider the CE for a massless scalar field subjected to Dirichlet b.c. on a sphere in the light of the above developed theory. So, following [9] we will consider the original sphere, of radius a, embedded in a concentric greater sphere of radius $R > a$. Then, the inner Casimir problem corresponds to the region $K_L \equiv K_a = \{\vec{x}|\ |\vec{x}|^2 \leq a^2\}$ and the outer problem corresponds to the region $K_R \backslash K_a = \{\vec{x}|\ a^2 \leq |\vec{x}|^2 \leq R^2\}$. However, here it is not convenient to consider these regions separately because, as already mentioned, there are logarithmic contributions for the

CE both from the inner and outer regions; only the sum is free of such terms (see, e.g, [29]). So, taking into account the $(2l+1)$-fold degeneracy of each eigenvalue, we have

$$E_{\rm vac} = \sum_{l=0}^{\infty}(l+\frac{1}{2})\sum_{n=1}^{\infty}\omega_{nl}\, C_{nl}(\Lambda)\ , \tag{89}$$

where the eigenfrequencies are given by

$$f_l^{(1)}(\omega a) \equiv j_l(\omega a) = 0\ , \tag{90a}$$

for the inner region (K_a), and by [9]

$$f_l^{(2)}(\omega a) \equiv j_l(\omega a) + \tan\delta_l(\omega a)\ n_l(\omega a) = 0\ , \tag{90b}$$

with

$$\tan\delta_l(\omega a) = (\omega a)\frac{R}{a} - \frac{l\pi}{2}\ , \tag{90c}$$

for the outer one $(K_R \backslash K_a)$. In these expressions j_l and n_l are the spherical Bessel functions. Notice that (89) corresponds to the general case (1) where $\alpha \equiv (n,l)$.

The calculation of the CE for a sphere using mode-by-mode summation was enormously simplified after the work of Nesterenko and Pirozhenko [8] who have made use of the Cauchy theorem for change the n-sum in (89) into an integral. This method was additionally developed in [9], where an exponential cutoff (which is more physical) was used. Here we will use the method of [9] with an important difference: the cutoff functions used in [9], while appropriate to treat the eletromagnetic field, do not render the integrals well-defined in the massless scalar field case, so that we will adopt

$$C_{nl}(\Lambda) = e^{-\Lambda(\nu/a+\omega_{nl})}\ , \tag{91}$$

for the cutoff functions, which satisfies the normalization condition (2). Besides this, it is important to analyse the asymptotic behaviour of $E_{\rm vac}$ based on more general cutoff functions.

Then, in the notation of refs. [8, 9], by using the Cauchy theorem we can rewrite (89) as

$$E_{\rm vac} = -\frac{1}{a}\sum_{l=0}^{\infty} Q_l\ , \tag{92}$$

where $(\nu = l + \frac{1}{2})$

$$Q_l \equiv \frac{\nu}{\pi}e^{-\Lambda\nu/a}{\rm Re}\, e^{-i\varphi}\int_0^{\infty} y\exp\{-i\frac{\Lambda}{a}ye^{-i\varphi}\}\frac{d}{dy}\ln f_l(iye^{-i\varphi})\ dy\ . \tag{93}$$

In this expression $\varphi \neq 0$ is an (small) angle which orientates the contour of integration with respect to the imaginary axis of z (see [9] – this very clever contour removes any dependence in R, so that we do not have to subtract an extensive term depending on R). The function f_l is defined as

$$f_l(z) = f_l^{(1)}(z) f_l^{(2)}(z) \,, \tag{94}$$

where $f_l^{(1)}(z)$ is given by (90a) and $f_l^{(2)}(z)$ is obtained from (90b) noting that on the relevant part of the integration contour $\tan\delta_l(z) \to i$ (or $-i$) [9], so that

$$f_l^{(2)}(z) = j_l(z) + i n_l(z) = h_l^{(1)}(z) \,, \tag{95}$$

where $h_l^{(1)}$ is the spherical Bessel function of the third kind (Hankel's function). Then, using

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_\nu(z) \,; \tag{96a}$$

$$h_l^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H_\nu^{(1)}(z) \,, \tag{96b}$$

and the fact that $J_\nu(iz) = i^\nu I_\nu(z)$ and $K_\nu(z) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(iz)$, where I_ν and K_ν are the modified Bessel functions, we obtain

$$f_l(iye^{-i\varphi}) = -\frac{1}{ye^{-i\varphi}} I_\nu(ye^{-i\varphi}) K_\nu(ye^{-i\varphi}) \,. \tag{97}$$

The expression (93) may be written in a convenient way reescaling $y \to \nu y$, when we obtain

$$Q_l = \frac{\nu^2}{\pi} e^{-\Lambda\nu/a} \mathrm{Re}\, e^{-i\varphi} \int_0^\infty y \exp\{-i\nu \frac{\Lambda}{a} y e^{-i\varphi}\} \frac{d}{dy} \ln f_l(i\nu y e^{-i\varphi})\, dy \,, \tag{98}$$

and using the uniform asymptotic expansions for I_ν and K_ν [31] (also see Appendix B) we can obtain an asymptotic expansion for Q_l which is valid for large orders. Then, in general, we can rewritte E_{vac} as [8]

$$E_{\mathrm{vac}} = -\frac{1}{a} \left(\sum_{l=0}^{n} \Delta Q_l + \sum_{l=0}^{\infty} Q_l^{\mathrm{asym}} \right) \,, \tag{99}$$

where Q_l^{asym} stands for the expression obtained from (98) by using the asymptotic expansions (B.1a) and (B.1b) for the Bessel functions, $\Delta Q_l = Q_l - Q_l^{\mathrm{asym}}$ and n is such that for $l > n$ the asymptotic expansion Q_l^{asym} affords a good approximation for Q_l (i.e., $\Delta Q_l \simeq 0$ for $l > n$).

Lets now to calculate the last term in (99). In order to proceed, it is convenient to define

$$E_{\mathrm{asym}} = -\frac{1}{a} \sum_{l=0}^{\infty} Q_l^{\mathrm{asym}} \,, \tag{100}$$

where, by using (B.4), Q_l^{asym} may be written as

$$
\begin{aligned}
Q_l^{\text{asym}} &= \frac{\nu^2}{\pi}\text{Re } e^{-i\varphi}\int_0^\infty y\exp\{-\frac{\nu\Lambda}{a}[1+iye^{-i\varphi}]\} \\
&\times \frac{d}{dy}\ln\left\{-\frac{1}{2\nu^2}\frac{t}{ye^{-i\varphi}}\left[1+\frac{\alpha(t)}{\nu^2}+\frac{\beta(t)}{\nu^4}+\ldots\right]\right\} dy\ ,
\end{aligned}
\tag{101}
$$

with $t = (1+y^2e^{-2i\varphi})^{\frac{1}{2}}$ and $\alpha(t)$, $\beta(t)$ are defined in Appendix B, eqs. (B.5a) and (B.5b). Then, expanding in a Taylor series for $\ln(1+x)$ and substituting into (100), we get

$$
\begin{aligned}
E_{\text{asym}} &= -\frac{1}{\pi a}\sum_{l=0}^{\infty}\nu^2\text{Re } e^{-i\varphi}\int_0^\infty y\exp\{-\frac{\nu\Lambda}{a}[1+iye^{-i\varphi}]\}\frac{d}{dy}\ln t\ dy \\
&- \frac{1}{\pi a}\sum_{l=0}^{\infty}\text{Re } e^{-i\varphi}\int_0^\infty y\exp\{-\frac{\nu\Lambda}{a}[1+iye^{-i\varphi}]\}\frac{d}{dy}\alpha(t)\ dy \\
&- \frac{1}{\pi a}\sum_{l=0}^{\infty}\frac{1}{\nu^2}\text{Re } e^{-i\varphi}\int_0^\infty y\exp\{-\frac{\nu\Lambda}{a}[1+iye^{-i\varphi}]\} \\
&\times \frac{d}{dy}\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right] dy + \cdots\ .
\end{aligned}
\tag{102}
$$

The above equation may be additionally simplified by transforming positive powers of ν into derivatives of Λ and performing the l-sums in these terms, resulting

$$
\begin{aligned}
E_{\text{asym}} &= -\frac{a}{\pi}\frac{d^2}{d\Lambda^2}\text{Re } e^{-i\varphi}\int_0^\infty y\frac{\exp\{-\frac{\Lambda}{2a}[1+iye^{-i\varphi}]\}}{1-\exp\{-\frac{\Lambda}{a}[1+iye^{-i\varphi}]\}}\frac{1}{(1+iye^{-i\varphi})^2}\frac{d}{dy}\ln t\ dy \\
&- \frac{1}{\pi a}\text{Re } e^{-i\varphi}\int_0^\infty y\frac{\exp\{-\frac{\Lambda}{2a}[1+iye^{-i\varphi}]\}}{1-\exp\{-\frac{\Lambda}{a}[1+iye^{-i\varphi}]\}}\frac{d}{dy}\alpha(t)\ dy \\
&- \frac{1}{\pi a}\sum_{l=0}^{\infty}\frac{1}{\nu^2}\text{Re } e^{-i\varphi}\int_0^\infty y\exp\{-\frac{\nu\Lambda}{a}[1+iye^{-i\varphi}]\} \\
&\times \frac{d}{dy}\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right] dy + \cdots\ .
\end{aligned}
\tag{103}
$$

Now we may expand the Λ-dependent terms in Taylor series, so that E_{asym} is given by

$$
E_{\text{asym}} = -\frac{2}{\pi}\frac{a^2}{\Lambda^3}\text{Re } e^{-i\varphi}\int_0^\infty \frac{y}{(1+iye^{-i\varphi})^3}\frac{d}{dy}\ln t\ dy
$$

$$
\begin{aligned}
& - \frac{1}{\pi}\frac{1}{\Lambda}\mathrm{Re}\, e^{-i\varphi}\int_0^\infty \frac{y}{(1+iye^{-i\varphi})}\frac{d}{dy}\alpha(t)\, dy \qquad (104)\\
& - \frac{1}{\pi a}\zeta(2,\frac{1}{2})\mathrm{Re}\, e^{-i\varphi}\int_0^\infty y\frac{d}{dy}\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right] dy + \mathcal{O}(\Lambda)\ ,
\end{aligned}
$$

where we have made use of the definition of the Hurwitz zeta function:

$$
\zeta(s,a) = \sum_{k=0}^{\infty}(k+a)^{-s}\ , \tag{105}
$$

in such way that $\sum_{l=0}^{\infty}\frac{1}{\nu^2} = \zeta(2,\frac{1}{2})$.

From the eq. (104) must be clear the reason why we have introduced the cutoff functions (91) rather than $e^{-\Lambda\omega_{nl}}$ used in [9]. Namely, in the absence of the term $e^{-\Lambda\nu/a}$ in (91) the first integral in (104) would have a non-integrable singularity in the origin, but all integrals are well-defined if we adopt (91). Then we may perform a rotation of the integration contour ($ye^{-i\varphi}\to y$) obtaining

$$
\begin{aligned}
E_{\rm asym} = & -\frac{2}{\pi}\frac{a^2}{\Lambda^3}\mathrm{Re}\int_0^\infty \frac{y}{(1+iy)^3}\frac{d}{dy}\ln t\, dy \\
& - \frac{1}{\pi}\frac{1}{\Lambda}\mathrm{Re}\int_0^\infty \frac{y}{(1+iy)}\frac{d}{dy}\alpha(t)\, dy \qquad (106)\\
& - \frac{1}{\pi a}\zeta(2,\frac{1}{2})\ \mathrm{Re}\int_0^\infty y\frac{d}{dy}\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right] dy + \mathcal{O}(\Lambda)\ ,
\end{aligned}
$$

which finally yields

$$
E_{\rm asym} = -\frac{a^2}{8\,\Lambda^3} - \frac{5}{1024\,\Lambda} + \frac{35\pi^2}{65536\,a} + \mathcal{O}(\Lambda)\ , \tag{107}
$$

where we have used $\zeta(2,\frac{1}{2}) = \frac{\pi^2}{2}$.

Now, it remains to calculate $\sum_{l=0}^{n}\Delta Q_l$ in (99). Notice that in this term the sum is finite and we do not have any divergence. Then, since $\varphi>0$ may be considered a small angle ($\sin\varphi>0$ and $\cos\varphi>0$) we may integrate (93) by parts and manipulate to obtain

$$
Q_l \ = \ -\frac{\nu}{\pi}e^{-\Lambda\nu/a}\mathrm{Re}\int_0^\infty dy\left(1-i\frac{\Lambda}{a}y\right)\exp\{-i\frac{\Lambda}{a}y\}\ln\left[2yI_\nu(y)K_\nu(y)\right]\ , \tag{108}
$$

after performing a rotation of the integration contour $(ye^{-i\varphi}\to y)$. So, expanding in powers of Λ and realizing that the Λ-independent term is real, we get

$$
Q_l = -\frac{\nu}{\pi}\int_0^\infty dy\ln\left[2yI_\nu(yK_\nu(y)\right] + \mathcal{O}(\Lambda)\ , \tag{109}
$$

which is nothing but the Q_l in ref. [8] (except for a sign). So we may take advantage of the numerical results in [8] for this expression (see table I).

Analogously, we may obtain a expression for $Q_l^{\rm asym}$, appropriate for when there is no infinite summation, by performing the same manipulations in (101). Then we have

$$
\begin{aligned}
Q_l^{\rm asym} &= -\frac{\nu^2}{\pi}\int_0^\infty dy \frac{d}{dy}\ln\left[\frac{y}{\sqrt{1+y^2}}\right] - \frac{1}{\pi}\int_0^\infty dy\;\alpha(t) \\
&- \;\; \frac{1}{\pi\nu^2}\int_0^\infty dy\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right] + \mathcal{O}(\Lambda)\;, \qquad (110)
\end{aligned}
$$

which after integration yields

$$
Q_l^{\rm asym} = \frac{\nu^2}{2} + \frac{1}{128} - \frac{35}{32768\;\nu^2} + \cdots\;. \qquad (111)
$$

Then, from the table I must be clear that we can take $n = 4$ in (99), so obtaining

$$
E_{\rm vac} = -\frac{a^2}{8\Lambda^3} - \frac{5}{1024\Lambda} + \frac{0.002819}{a} + \cdots\;, \qquad (112)
$$

which yields $a_4^{\rm sphere} \simeq 0.002819$ for the coefficient of the Λ-independent term in the asymptotic series (3) for E. Therefore the CE is

$$
E_{\rm Casimir} = \frac{a_4^{\rm sphere}}{a} \simeq \frac{0.002819}{a}\;, \qquad (113)
$$

and by (24) we see that the Casimir force for massless scalar field with Dirichlet b.c. on a sphere is repulsive, in accordance with the results obtained in refs. [8, 9, 26].

l	ΔQ_l
0	0.002360
1	0.000076
2	0.000012
3	0.000003
4	0.000001

Table I: $\Delta Q_l = Q_l - Q_l^{\rm asym}$ with Q_l given by (109) and $Q_l^{\rm asym}$ by (111) (we have used the numerical results of ref. [8] for Q_l).

Some comments about the result (113) are noteworthy. First, although (113) is exactly the same result obtained in [8] and, in fact, we have managed to partially use the numerical results in this reference, there are important differences between the methods. Namely, the the regularization here used is more physical in the sense that it simulates the transparency of any material for large frequencies. In addition we have incorporated the correct b.c. at infinity allowing for reality of the eigenvalues, according to [9]. Secondly, the modification of the cutoff functions (91) when compared with [9] is crucial to obtain an everywhere well-defined expression for the energy. Finally, and most importantly, from the earlier discussion about the method here employed we claim that the Casimir energy (113) is independent of regularization (i.e., independent of the set $\{C_{nl}(\Lambda)\}$ adopted) and this is the most interesting point, along with the asymptotic series (112).

5 Conclusion and Open Problems

In this paper we have shown how Dietz's idea leads to a complete theory of the CE for compact regions. The CE is given as a (divergent) asymptotic series in two variables, the linear dimension of the region and an ultraviolet cutoff. The cutoff-independent term is the Ramanujan sum of the divergent series and, as such, independent of the type of ultraviolet cutoff function, except for its value at a fixed point. In particular, as proved in section 2 for parallel plates, it is not necessary to assume vanishing of the derivatives of the cutoff-function at this point, as done in the otherwise careful treatment of [18]. This would exclude the very useful exponential-type cutoffs used in sections 2, 3 and 4. We emphasize that there are no ill-defined limits or other dubious mathematical manipulations in this theory. Finally, RI is an essential requirement, which is not present in other approaches, such as the otherwise natural $\vec{x}$-space approach, as demonstrated in section 1. This is a defect due to an incomplete treatment of the surface: in a microscopic theory, a formidable problem which goes far beyond the present adoption of classical boundary conditions, both the $\vec{x}$-space method and the mode approach must, of course, match.

The exact result for the sphere in the previous section agrees with [8, 9] and leads to a *repulsive* force. Although references [8] and [9] have clarified major points of the theory, and greatly simplified Boyer's tough calculation [23], several infinities are disposed in an ad-hoc manner. The advantage of our treatment is that we succeeded in deriving a well-defined asymptotic series for the CE (112), upon introduction of a special cutoff. As already discussed, the cutoff-independent term does not depend on this special choice, the same happening in the case of the cube, where the exponential cutoff (17) played an important role in the explicit computation. The sphere's calculation still entails, however, some misterious features, already present in (22), or in our prescription of a contour (following [9]): the result (24) obtained is already the *local* pressure, and no divergence due to extensivity in the outer problem forces us to use (23c). In contrast, the completely analytic and rigorous solution obtained for the cube is entirely transparent and, as remarked in section 3, displays clearly the source of the repulsive interaction: the "edge" terms, which are due to the *curvature*. While for parallel plates the outer problem is of the same nature of the inner one, leading to vanishing contribution to the CE, both for the cube and for the sphere curvature effects change the computation of the outer problem drastically. For the sphere these effects appear, however, more indirectly, reflecting themselves in the appearance of the Neumann functions. In this context we refer however to [30] for a rigorous proof of the mode sum for the ball, showing why subtraction of the outer region of the sphere is enough to ensure convergence of the mode sum.

One may argue that our solution of the cube's outer problem involved the introduction of additional stresses in the region $K_M \backslash K_L$. The important point is that no additional stresses on the boundary of K_L are introduced, and, therefore, ours is a possible solution of the outer problem, because only the local density (i.e., at the boundary of K_L) is accesible to experiment. That is incidentally, the reason why there is no "obvious" b.c. at infinity for the Casimir problem. It is therefore expected that the CE as computed is independent of the b.c. placed on the "larger" region, but no general proof of this fact exists at present which goes beyond comparing exact calculations with different b.c.. It remains also, of course, as a difficult problem, to find a solution of the cube's outer problem without introducing additional stresses, and to prove that the result remains unaltered.

A very preliminary step in the explanation of the major role played by the curvature was already performed in [4], but, apparently, with other objectives in mind. Let the coefficients of the asymptotic series of $P(\vec{x},\vec{x};\Lambda)$, given by (19b) with $(-\Delta)^{\frac{1}{2}}$ replaced by $(-\Delta)$ be defined by:

$$P(\vec{x},\vec{x};\Lambda) = (4\pi)^{-\frac{d}{2}} \sum_{k=0}^{\infty} \Lambda^{(k-d)/2} C_k(\vec{x},\vec{x}) \ . \tag{114}$$

Due to (19a) and (114), $E_{\text{vac}}(\Lambda)$, with $(-\Delta)^{\frac{1}{2}}$ replaced by $(-\Delta)$, is an asymptotic series with coefficients $\Lambda^{[(k-d)/2-1]}$ and, thus, the Λ-independent term in the series for $E_{\text{vac}}(\Lambda)$ corresponding to (114) is given by $k = d+2$, hence $k = 5$ for $d = 3$. Unfortunately $C_5(\vec{x},\vec{x})$ does not seem to have been computed for compact manifolds with boundary (see [4] and references given there), but some of the coefficients available [4] indicate that the (Riemannian) curvature (or combinations of quantities involving the Riemannian, the mean and the Gaussian curvatures) may dominate in the calculation of $E_{\text{vac}}(\Lambda)$, while it is conceivable that, when the curvatures are zero negative terms dominate in $E_{\text{vac}}(\Lambda)$. In order to prove such a conjecture (under suitable – to be found – conditions on the manifold) it would be necessary to considerably extend the work of [4]. Two directions are needed: replacement of $(-\Delta)$ by $(-\Delta)^{\frac{1}{2}}$, and finding analytical methods to compute higher order coefficients. If, however, a conjecture of this type proved to be true, it would contribute to a geometric understanding of this intriguing issue. It could as well explain the very elusive caracter of the sign of the Casimir effect: since there is a competition between curvature and boundary terms, it is not obvious whether the resulting force is attractive or repulsive.

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Appendix A

In this appendix we present some aspects of the derivation of (25) not given in ([14]) which may arise doubts. The first term in (25) is the fundamental solution G of $(-\Delta)^{\frac{1}{2}}G = 2\delta$ ([6]. p. 341). The fundamental solution of $(-\Delta)^{\frac{1}{2}}G = 2\delta$ satisfying Dirichlet b.c. is of the form

$$G'(\vec{x},\vec{y}) = G(\vec{x},\vec{y}) + h(\vec{x},\vec{y}) \quad \vec{x},\ \vec{y}\ \in K_i\ , \tag{A.1}$$

where K_i denotes the interior of K, such that

$$(-\Delta)^{\frac{1}{2}}h(\vec{x},\vec{y}) = 0 \quad \vec{x},\ \vec{y}\ \in K_i\ , \tag{A.2}$$

and such that

$$G(\vec{x},\vec{y}) + h(\vec{x},\vec{y}) = 0\ , \tag{A.3}$$

where $\vec{x} \in K_i$ and $\vec{y} \in \partial K$, the boundary of K.

The second term in (25) does satisfy (A.2) because, if $\vec{x} \in K_i$, $\vec{y} \in K_e$ and vice-versa, where K_e denotes the exterior of K. Finally, (A.3) holds because, by the usual image method

$$\frac{1}{|\vec{x}-\vec{y}|} - \frac{R}{|\vec{y}|}\frac{1}{|\vec{x}-\vec{y}_*|} = 0\ , \tag{A.4}$$

whenever $\vec{x} \in K_i$ and $\vec{y} \in \partial K$. By (A.4)

$$\frac{1}{|\vec{x}-\vec{y}|^2} = \frac{R^2}{|\vec{y}|^2}\frac{1}{|\vec{x}-\vec{y}_*|^2}\ , \tag{A.5}$$

under the same conditions. Addition of $i0$ to the denominators corresponds to the usual principal value prescription for generalized functions[6] and (25) results.

Appendix B

In this appendix we list some asymptotic formulae for the modified Bessel functions which are necessary in the calculation of the CE in a spherical shell. The uniform asymptotic expansions are [31]

$$I_\nu(\nu z) \rightarrow \frac{1}{(2\pi\nu)^{\frac{1}{2}}}\frac{e^{\nu\eta(z)}}{(1+z^2)^{\frac{1}{4}}}\left\{1 + \sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\right\}\ ; \tag{B.1a}$$

$$K_\nu(\nu z) \rightarrow \frac{\pi}{(2\pi\nu)^{\frac{1}{2}}}\frac{e^{-\nu\eta(z)}}{(1+z^2)^{\frac{1}{4}}}\left\{1 + \sum_{k=1}^{\infty}(-1)^k\frac{u_k(t)}{\nu^k}\right\}\ , \tag{B.1b}$$

where

$$t = \sqrt{1+z^2} \ , \tag{B.2a}$$

$$\eta(z) = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}} \ , \tag{B.2b}$$

and the u_k functions are

$$u_1(t) \ = \ \frac{1}{24}\left(3t - 5t^3\right) \ ; \tag{B.3a}$$

$$u_2(t) \ = \ \frac{1}{1152}\left(81t^2 - 462t^4 + 385t^6\right) \ ; \tag{B.3b}$$

$$u_3(t) \ = \ \frac{1}{414720}\left(30375t^3 - 369603t^5 + 765765t^7 - 425425t^9\right) \ ; \tag{B.3c}$$

$$\begin{aligned} u_4(t) \ &= \ \frac{1}{39813120}\left(4465125t^4 - 94121676t^6\right. \\ &+ \left. 349922430t^8 - 446185740t^{10} + 185910725t^{12}\right) \ . \end{aligned} \tag{B.3d}$$

Then, for the product $I_\nu K_\nu$ we have

$$I_\nu(\nu z)K_\nu(\nu z) \simeq \frac{t}{2\nu}\left\{1 + \frac{\alpha(t)}{\nu^2} + \frac{\beta(t)}{\nu^4} + \ldots\right\} \ , \tag{B.4}$$

with $\alpha(t)$ and $\beta(t)$ defined as

$$\alpha(t) \equiv 2u_2(t) - u_1^2(t) \ ; \tag{B.5a}$$

$$\beta(t) \equiv 2u_4(t) - 2u_1(t)u_3(t) + u_2^2(t) \ . \tag{B.5b}$$

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